

STRUCTURE OF FINITE DIHEDRAL GROUP ALGEBRA

F. E. BROCHERO MARTÍNEZ

ABSTRACT. In this article, we show the relation between the irreducible idempotents of the cyclic group algebra $\mathbb{F}_q C_n$ and the central irreducible idempotents of the group algebras $\mathbb{F}_q D_{2n}$, where \mathbb{F}_q is a finite field with q elements and D_{2n} is the dihedral group of order $2n$, where $\gcd(q, n) = 1$.

In addition, if every divisor of n divides $q - 1$, we show explicitly all central irreducible idempotents of this group algebra and its Wedderburn decomposition.

1. INTRODUCTION

Let K be a field and G be a group with n elements. It is known that, if $\text{char}(K) \nmid n$, then the group algebra KG is semisimple and as consequence of Wedderburn Theorem, we have that KG is isomorphic to a direct sum of matrix algebras over division rings, such that each division algebra is a finite algebra over the field K , i.e, there exists an isomorphism

$$\rho : KG \rightarrow M_{l_1}(D_1) \oplus M_{l_2}(D_2) \oplus \cdots \oplus M_{l_t}(D_t),$$

where D_j are division rings such that $|G| = \sum_{j=1}^t l_j^2 [D_j : K]$. Observe that KG has t central irreducible idempotents, each one of the form

$$e_i = \rho^{-1}(0 \oplus \cdots \oplus 0 \oplus I_i \oplus 0 \cdots \oplus 0),$$

where I_i is the identity matrix of the component $M_{l_i}(D_j)$. Then, the isomorphism ρ determines explicitly each central irreducible idempotent.

In the case $K = \mathbb{Q}$, the calculus of central idempotents and Wedderburn decomposition is widely studied; the classical method to calculate the primitive central idempotents of group algebras depends on computing the character group table. Other method is shown in [8], where Jespers, Leal and Paques describe the central irreducible idempotents when G is a nilpotent group, using the structure of its subgroups, without employing the characters of the group. Generalizations and improvements of this method can be found in [11], where the authors provide information about the Wedderburn decomposition of $\mathbb{Q}G$. This computational method is also used in [2] to compute the Wedderburn decomposition and the primitive central idempotents of a semisimple finite group algebra KG , where G is an abelian-by-supersolvable group G and K is a finite field.

The structure of KG when $G = D_{2n}$ is the dihedral group with $2n$ elements is well known for $K = \mathbb{Q}$ (see [7]). In [5], Dutra, Ferraz and Polcino Milies impose conditions over q and n in order for $\mathbb{F}_q D_{2n}$ to have the same number of irreducible components that $\mathbb{Q}D_{2n}$. This result is generalized in [6], where Ferraz Goodaire

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and Polcino Milies find, for some families of groups, conditions under q and G in order for $\mathbb{F}_q G$ to have the minimum number of simple components.

In this article, assuming that every prime factor of n divides $q - 1$, we show explicitly the central irreducible idempotents of $\mathbb{F}_q D_{2n}$ and an isomorphism between the group algebra $\mathbb{F}_q D_{2n}$ and its Wedderburn decomposition. Observe that this isomorphism also show the structure of $\mathcal{U}(\mathbb{F}_q D_{2n})$, the unit group of $\mathbb{F}_q D_{2n}$.

2. IDEMPOTENTS OF CYCLIC GROUP ALGEBRA

Throughout this article, \mathbb{F}_q denotes a finite field of order q , where q is a power of a prime and n is a positive integer such that $\gcd(n, q) = 1$. For every polynomial $g(x)$ with $g(0) \neq 0$, g^* denotes the *reciprocal polynomial* of g , i.e., $g^*(x) = x^{\deg(g)} g(\frac{1}{x})$. The polynomial $x^n - 1 \in \mathbb{F}_q[x]$ splits in monic irreducible factors as

$$x^n - 1 = f_1 f_2 \cdots f_r f_{r+1} f_{r+1}^* f_{r+2} f_{r+2}^* \cdots f_{r+s} f_{r+s}^*,$$

where $f_1 = x - 1$, $f_2 = x + 1$ if n is even, and $f_j^* = f_j$ for $2 \leq j \leq r$, where r is the number of auto-reciprocal factors in the factorization and $2s$ the number of non-auto-reciprocal factors.

We denote by C_n the cyclic group of order n . It is well known that $\mathbb{F}_q C_n \simeq \mathcal{R}_n = \frac{\mathbb{F}_q[x]}{\langle x^n - 1 \rangle}$, and by the Chinese Remainder Theorem

$$\frac{\mathbb{F}_q[x]}{\langle x^n - 1 \rangle} \simeq \bigoplus_{j=1}^{r+s} \frac{\mathbb{F}_q[x]}{\langle f_j \rangle} \oplus \bigoplus_{j=r+1}^{r+s} \frac{\mathbb{F}_q[x]}{\langle f_j^* \rangle}$$

is exactly the Wedderburn decomposition of the group algebra \mathcal{R}_n , so every primitive idempotent generates a maximal ideal of \mathcal{R}_n and also one component of this direct sum.

In addition, since \mathcal{R}_n is a principal ideal domain, every ideal of \mathcal{R}_n is generated by a polynomial g that is a divisor of $x^n - 1$. The relation between the generator of the ideal and its principal idempotent is shown in the following lemma.

Lemma 2.1. *Let $\mathcal{I} \subset \mathcal{R}_n$ be an ideal generated by the monic polynomial g , that is divisor of $x^n - 1$, and define $f = \frac{x^n - 1}{g}$. Then the principal idempotent of \mathcal{I} is*

$$e_f = -\frac{((f^*)')^*}{n} \cdot \frac{x^n - 1}{f}.$$

Proof: Let t be an integer such that n divides $q^t - 1$. By Theorem 2.1 in [1] (see also Theorem 3.4 in [3]), every primitive idempotent of $\frac{\mathbb{F}_t[x]}{\langle x^n - 1 \rangle}$ is given by

$$u_\lambda = \frac{\lambda}{n} \cdot \frac{x^n - 1}{x - \lambda} = \frac{1}{n} \sum_{l=0}^{n-1} \lambda^{-l} x^l$$

where $\lambda^n = 1$.

Since f divides $x^n - 1$, then f splits in $\mathbb{F}_{q^t}[x]$ as $(x - \lambda_1) \cdots (x - \lambda_k)$ and

$$(f^*)' = \sum_{i=1}^k (-\lambda_i) \prod_{i \neq j} (1 - \lambda_j x) = f^* \sum_{i=1}^k \frac{-\lambda_i}{x - \lambda_j x},$$

hence

$$e_f = -\frac{((f^*)')^*}{n} \cdot \frac{x^n - 1}{f} = \sum_{i=1}^k \frac{\lambda_i}{n} \cdot \frac{x^n - 1}{x - \lambda} = \sum_{i=1}^k u_{\lambda_i}.$$

Therefore e_f is an idempotent of $\mathbb{F}_q[x]$. In order to prove that e_f is the principal idempotent of \mathcal{I} , it is enough to show that $g \cdot e_f = g$. Observe that, using partial fraction decomposition we obtain

$$g = \frac{x^n - 1}{f} = \sum_{i=1}^k A_i u_{\lambda_i},$$

where $A_i = \frac{n}{\lambda_i} \frac{1}{\prod_{j \neq i} (\lambda_i - \lambda_j)}$ and then

$$g \cdot e_f = \sum_{i=1}^k A_i u_{\lambda_i} \cdot \sum_{j=1}^k u_{\lambda_j} = \sum_{1 \leq i, j \leq k} A_i u_{\lambda_i} u_{\lambda_j} = \sum_{i=1}^k A_i u_{\lambda_i} = g$$

as we wanted to prove. \square

Remark 2.2. *This lemma is also true for fields with characteristic zero, it suffices to change in the proof the field \mathbb{F}_q by the splitting field of the polynomial f .*

Corollary 2.3. *The cyclic group ring \mathcal{R}_n has $r + 2s$ irreducible idempotents of the form e_f given by Lemma 2.1, where the polynomials f 's are the irreducible factors of $x^n - 1 \in \mathbb{F}_q[x]$.*

3. CENTRAL IDEMPOTENTS OF DIHEDRAL GROUP ALGEBRA

Throughout this section, α_j denotes a root of the polynomial f_j and D_{2n} denotes the dihedral group of order $2n$, i.e.

$$D_{2n} = \langle x, y | x^n = 1, y^2 = 1, xy = yx^{-1} \rangle.$$

We define integer numbers ϵ and δ as

$$\epsilon = \begin{cases} 0 & \text{if } \text{char}(q) = 2 \\ 1 & \text{if } \text{char}(q) \neq 2 \text{ and } n \text{ is odd} \\ 2 & \text{if } \text{char}(q) \neq 2 \text{ and } n \text{ is even} \end{cases}$$

and $\delta = \max\{\epsilon, 1\}$.

The following theorem shows explicitly the dependence of the Wedderburn decomposition of the Dihedral group algebra over a finite field \mathbb{F}_q with the factorization of $x^n - 1 \in \mathbb{F}_q[x]$.

Theorem 3.1. *The group algebra $\mathbb{F}_q D_{2n}$ has Wedderburn decomposition of the form*

$$\mathbb{F}_q D_{2n} \cong \bigoplus_{j=1}^{r+s} A_j$$

where

$$A_j = \begin{cases} \mathbb{F}_q \oplus \mathbb{F}_q & \text{if } j \leq \delta, \\ M_2(\mathbb{F}_q[\alpha_j + \alpha_j^{-1}]) & \text{if } \delta + 1 \leq j \leq r, \\ M_2(\mathbb{F}_q[\alpha_j]) & \text{if } r + 1 \leq j \leq r + s, \end{cases}$$

Proof: For each $j \in \{1, \dots, s + r\}$, let τ_j be the homomorphism of \mathbb{F}_q -algebras defined by the generators of the group D_{2n} as

$$\begin{aligned} \tau_1 : \mathbb{F}_q D_{2n} &\rightarrow \mathbb{F}_q \oplus \mathbb{F}_q \\ x &\mapsto (1, 1) \\ y &\mapsto (1, -1), \end{aligned}$$

in the case $\epsilon \geq 1$ and

$$\begin{aligned} \tau_2 : \mathbb{F}_q D_{2n} &\rightarrow \mathbb{F}_q \oplus \mathbb{F}_q \\ x &\mapsto (-1, -1) \\ y &\mapsto (1, -1), \end{aligned}$$

in the case $\epsilon = 2$, where the sum and product in $\mathbb{F}_q \oplus \mathbb{F}_q$ is defined by adding and multiplying the corresponding components of the same coordinates. Finally, for every $j \geq \epsilon + 1$

$$\begin{aligned} \tau_j : \mathbb{F}_q D_{2n} &\rightarrow M_2(\mathbb{F}_q[\alpha_j]) \\ x &\mapsto \begin{pmatrix} \alpha_j & 0 \\ 0 & \alpha_j^{-1} \end{pmatrix} \\ y &\mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

It is easy to prove that $(\tau_j(x))^n = I$, $(\tau_j(y))^2 = I$ and $\tau_j(x)\tau_j(y) = \tau_j(y)\tau_j(x)^{-1}$.

Observe that in the case of characteristic 2, i.e. $\epsilon = 0$, we have that

$$\text{img}(\tau_1) = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \middle| a, b \in \mathbb{F}_q \right\},$$

that is isomorphic to $\mathbb{F}_q \oplus \mathbb{F}_q$ by the projection σ_0 over the first row of the matrix, where the product is defined by $(a, b) \cdot (c, d) = (ac + bd, ad + bc)$. Thus, $\dim_{\mathbb{F}_q}(\text{img}(\tau_1)) = 2$ in all cases. In addition, if n is even, then $\dim_{\mathbb{F}_q}(\text{img}(\tau_2)) = 2$.

For each $\delta < j \leq r$, if we define $Z_j = \begin{pmatrix} 1 & -\alpha_j \\ 1 & -\alpha_j^{-1} \end{pmatrix}$, then

$$\begin{aligned} \sigma_j : M_2(\mathbb{F}_q[\alpha_j]) &\rightarrow M_2(\mathbb{F}_q[\alpha_j]) \\ X &\mapsto Z_j^{-1} X Z_j \end{aligned}$$

is an automorphism such that

$$\sigma_j \circ \tau_j(x) = \begin{pmatrix} 0 & 1 \\ -1 & \alpha_j + \alpha_j^{-1} \end{pmatrix} \quad \text{and} \quad \sigma_j \circ \tau_j(y) = \begin{pmatrix} 1 & -(\alpha_j + \alpha_j^{-1}) \\ 0 & -1 \end{pmatrix},$$

so the images of the generators of D_n are in $\mathbb{F}_q(\alpha_j + \alpha_j^{-1})$. It follows that for each j such that $\delta < j \leq r$ we have

$$\dim_{\mathbb{F}_q}(\text{img}(\tau_j)) = \dim_{\mathbb{F}_q}(\text{img}(\sigma_j \circ \tau_j)) \leq 4 \dim_{\mathbb{F}_q}(\mathbb{F}_q(\alpha_j + \alpha_j^{-1})) = 2 \deg(f_j)$$

and in the case $r + 1 \leq j \leq r + s$, we know that

$$\dim_{\mathbb{F}_q} \text{img}(\tau_j) \leq 4 \dim_{\mathbb{F}_q} \mathbb{F}_q(\alpha_j) = 4 \deg(f_j).$$

Now, let τ be the homomorphism of \mathbb{F}_q -algebras defined by $\bigoplus_{j=1}^{s+r} \tau_j$. Observe that this homomorphism is injective. In fact, let u be an element of $\mathbb{F}_q D_n$ such that $\tau(u) = 0$. If we write $u = P_1(x) + P_2(x)y$, where P_1 and P_2 are polynomials of degree less than n , for each $j > \epsilon$, we have

$$\tau_j(u) = \begin{pmatrix} P_1(\alpha_j) & P_2(\alpha_j) \\ P_2(\alpha_j^{-1}) & P_1(\alpha_j^{-1}) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

so, $P_1(\alpha_j) = P_1(\alpha_j^{-1}) = 0$ and $P_2(\alpha_j) = P_2(\alpha_j^{-1}) = 0$. In addition, if $\epsilon \geq 1$, then

$$\tau_1(u) = (P_1(1) + P_2(1), P_1(1) - P_2(1)) = 0,$$

and if $\epsilon = 2$ we have

$$\tau_2(u) = (P_1(-1) + P_2(-1), P_1(-1) - P_2(-1)) = 0.$$

It follows that P_1 and P_2 are divisible by the polynomial $x^n - 1$ and since the degrees of these polynomials are less than n , we conclude that P_1 and P_2 are null polynomials and therefore τ is an injective homomorphism.

Finally, we observe that the homomorphism $\rho : \mathbb{F}_q D_{2n} \rightarrow \bigoplus_{j=1}^{r+s} A_j$ defined by $\rho = \bigoplus_{j=1}^{r+s} \rho_j$ where $\rho_j = \begin{cases} \sigma_j \circ \tau_j & \text{if } \epsilon < j \leq r \\ \tau_j & \text{otherwise} \end{cases}$ is injective. Furthermore, $\dim_{\mathbb{F}_q}(\mathbb{F}_q D_n) = 2n$ and

$$\begin{aligned} \dim_{\mathbb{F}_q} \left(\bigoplus_{j=1}^{r+s} A_j \right) &= 2\epsilon + 4 \sum_{j=\epsilon+1}^r \dim_{\mathbb{F}_q}(\mathbb{F}_q[\alpha_j + \alpha_j^{-1}]) + 4 \sum_{j=r+1}^{r+s} (\dim_{\mathbb{F}_q}(\mathbb{F}_q[\alpha_j])) \\ &= 2\epsilon + 2 \sum_{j=\epsilon+1}^r \deg(f_j) + 4 \sum_{j=r+1}^{r+s} \deg(f_j) \\ &= 2 \deg(x^n - 1) = 2n. \end{aligned}$$

Therefore ρ is an isomorphism. \square

Remark 3.2. In the proof of the theorem we use the following facts: if β is a root of the polynomial $g \in \mathbb{F}_q[x]$, then β^{-1} is root of the polynomial g^* . In addition, when g is auto-reciprocal and ± 1 are not roots of g , there exists a polynomial $h \in \mathbb{F}_q[x]$ of degree $\frac{\deg(g)}{2}$, such that β is a root of g if and only if $\beta + \beta^{-1}$ is a root of h . In fact, since g is symmetrical, we can write g as

$$g(x) = \sum_{j=0}^t a_j(x^{t+j} + x^{t-j}) = x^t \sum_{j=0}^t a_j(x^j + x^{-j}) = x^t \sum_{j=0}^t a_j D_j(z, 1) = x^t h(z)$$

where D_j is the Dickson polynomial of degree j and $z = x + x^{-1}$ (see [9] or [10]).

Theorem 3.3. The dihedral group algebra $\mathbb{F}_q D_{2n}$ has $\epsilon + r + s$ central irreducible idempotents:

- (1) 2ϵ idempotents of the form $\frac{1+y}{2}e_{f_j}$ and $\frac{1-y}{2}e_{f_j}$, where $j \leq \epsilon$.
- (2) $r-\epsilon$ idempotents e_{f_j} , where $j = \epsilon+1, \dots, r$, generated by the auto-reciprocals factor of $x^n - 1$.
- (3) s idempotents $e_{f_j} + e_{f_j^*}$, where $j = r+1, \dots, r+s$.

Proof: Since the homomorphism τ in the proof of Theorem 3.1 is injective, then the image of a central primitive idempotent u by the homomorphism has to be zero in every component, except for one component where the image is the identity, i.e., for some i fixed, $\tau_j(u) = \delta_{i,j} I_j$, where I_j is the identity over the component A_j . Let $u = P(x) + Q(x)y$ be a representation of u , where P and Q are polynomials in $\mathbb{F}_q[x]$ of degree less than or equal to $n-1$. Observe that Q is zero when calculated at each root of the polynomial $x^n - 1 = 0$, so Q is the null polynomial. In addition, P is one when we calculate it at the roots of the polynomials f_j and f_j^* and zero when we calculate it at the other roots of the polynomial $x^n - 1$. The unique polynomial of degree less or equal to $n-1$ that satisfies that proprieties is e_{f_j} , when $f_j = f_j^*$ and $e_{f_j} + e_{f_j^*}$ when $f_j \neq f_j^*$. Finally, if $j \leq \epsilon$ the image $\tau_j(e_{f_j}) = (1, 1)$ is not a primitive idempotent, and we can decompose this idempotent in two central primitive idempotents, $(\frac{1+y}{2})e_{f_j}$ and $(\frac{1-y}{2})e_{f_j}$, such that $\tau_j((\frac{1+y}{2})e_{f_j}) = (1, 0)$ and $\tau_j((\frac{1-y}{2})e_{f_j}) = (0, 1)$. \square

4. EXPLICIT FORM OF THE IDEMPOTENTS WHEN $\text{rad}(n)|(q-1)$

Throughout this section, we assume that every prime factor of n divides $q-1$, κ and ν denote the numbers $\gcd(n, q-1)$ and $\min\{\nu_2(\frac{n}{\kappa}), \nu_2(q-1)\}$ respectively, θ and α are generators of \mathbb{F}_q^* and $\mathbb{F}_{q^2}^*$ such that $\alpha^{q+1} = \theta$. In the following results, we show the explicit form of the idempotents of the cyclic group algebra $\mathbb{F}_q C_n$ and the Wedderburn decomposition of the Dihedral group algebra $\mathbb{F}_q D_{2n}$. In order to show that representation, we need the following lemma

Lemma 4.1. [4, Corollary 3.3 and Corollary 3.6] *The factorization of $x^n - 1$ in irreducible factors of $\mathbb{F}_q[x]$ depends on n and q in the following form:*

(i) *If $8 \nmid n$ or $q \not\equiv 3 \pmod{4}$, then*

$$x^n - 1 = \prod_{t|m} \prod_{\substack{1 \leq u \leq \gcd(n, q-1) \\ \gcd(u, t)=1}} (x^t - \theta^{ul}),$$

where $m = \frac{n}{\kappa}$ and $l = \frac{q-1}{\kappa}$. In addition, for each t such that $t|m$, the number of irreducible factors of degree t is $\frac{\varphi(t)}{t} \cdot \kappa$, where φ denotes the Euler Totient function.

(ii) *If $8 | n$ and $q \equiv 3 \pmod{4}$, then*

$$x^n - 1 = \prod_{\substack{t|m' \\ t \text{ odd}}} \prod_{\substack{1 \leq w \leq \kappa \\ \gcd(w, t)=1}} (x^t - \theta^{wl}) \cdot \prod_{t|m'} \prod_{u \in \mathcal{S}_t} (x^{2t} - (\alpha^{ul'} + \alpha^{qu l'})x^t + \theta^{ul'}),$$

where $m' = \frac{n}{2^\nu \kappa}$, $l' = \frac{q^2-1}{2^\nu \kappa}$, and \mathcal{S}_t is the set

$$\left\{ u \in \mathbb{N} \mid \begin{array}{l} 1 \leq u \leq 2^\nu \kappa, \gcd(u, t) = 1 \\ 2^\nu \nmid u \text{ and } u < \{qu\}_{2^\nu \kappa} \end{array} \right\},$$

where $\{a\}_b$ denotes the remainder of the division of a by b , i.e. the number $0 \leq c < b$ such that $a \equiv c \pmod{b}$. In addition, for each t odd such that $t|m'$, the number of irreducible binomials of degree t and $2t$ is $\frac{\kappa \cdot \varphi(t)}{t}$ and $\frac{\kappa \cdot \varphi(t)}{2t}$ respectively, and the number of irreducible trinomials of degree $2t$ is

$$\begin{cases} \frac{\varphi(t)}{t} \cdot 2^{\nu-1} \kappa, & \text{if } t \text{ is even} \\ \frac{\varphi(t)}{t} \cdot (2^{\nu-1} - 1) \kappa, & \text{if } t \text{ is odd.} \end{cases}$$

The following corollary, direct from Lemmas 2.1 and 4.1, shows the explicit form of each idempotent of the cyclic group algebra $\mathbb{F}_q C_n$ when $\text{rad}(n)|(q-1)$.

Corollary 4.2. *Let m , m' , l and l' be as in Lemma 4.1.*

(1) *If $8 \nmid n$ or $n \not\equiv 3 \pmod{4}$, then every irreducible idempotent of the ring \mathcal{R}_n is of the form*

$$e_{t,ul} = \frac{\theta^{ul} t}{n} \cdot \frac{x^n - 1}{x^t - \theta^{ul}}$$

where t and u satisfy the condition of Lemma 4.1 item (i).

(2) *If $8 | n$ or $n \equiv 3 \pmod{4}$, then every irreducible idempotent of the ring \mathcal{R}_n is of the form shown in (1) and of the form*

$$e_{t,ul'} = \frac{t}{n} \left((\alpha^{ul'} + \alpha^{ul'q})x^t - 2\theta^{ul'} \right) \frac{x^n - 1}{(x^{2t} - (\alpha^{ul'} + \alpha^{qu l'})x^t + \theta^{ul'})},$$

where t and u satisfy the condition of Lemma 4.1 item (ii).

Remark 4.3. By Theorem 3.3 If

- $\text{char}(\mathbb{F}_q) = 2$, or
- n is odd and $\theta^{ul} \neq 1$, or
- n is even and $\theta^{ul} \neq \pm 1$,

then every idempotent found in Corollary 4.2 item (1) is a central irreducible idempotent of $\mathbb{F}_q D_{2n}$. Otherwise, the idempotent can be reduced to two central primitive idempotents $\frac{1+y}{2}e_{t,ul}$ and $\frac{1-y}{2}e_{t,ul}$.

In addition, $e_{t,ul'}$ of item (2) is also a central irreducible idempotent of $\mathbb{F}_q D_{2n}$ if $\theta^{ul'} = 1$, otherwise, the central irreducible idempotent is $e_{t,ul'} + e_{t,-ul'}$.

Theorem 4.4. The Wedderburn decomposition of the group algebra $\mathbb{F}_q D_{2n}$ depends on n and q in the following form:

(1) When n is odd, the decomposition is

$$2\mathbb{F}_q \oplus \frac{\kappa-1}{2}M_2(\mathbb{F}_q) \oplus \bigoplus_{\substack{t|m \\ t \neq 1}} \frac{\kappa \cdot \varphi(t)}{2t} M_2(\mathbb{F}_{q^t}).$$

(2) When n is even,

(2.1) if $q \equiv 1 \pmod{4}$ or $8 \nmid n$, the decomposition is

$$4\mathbb{F}_q \oplus \left(\frac{\kappa}{2} - 1\right) M_2(\mathbb{F}_q) \oplus \bigoplus_{\substack{t|m \\ t \neq 1}} \frac{\kappa \cdot \varphi(t)}{2t} M_2(\mathbb{F}_{q^t}),$$

(2.2) if $q \equiv 3 \pmod{4}$ and $8|n$, the decomposition is

$$\begin{aligned} & 4\mathbb{F}_q \oplus (\kappa + 2^{\nu-i} - 3)M_2(\mathbb{F}_q) \oplus (2^{\nu-2}\kappa - 2^{\nu-1} - \frac{k}{4} + 1)M_2(\mathbb{F}_{q^2}) \oplus \bigoplus_{\substack{t|m' \\ t \text{ odd} \\ t \neq 1}} \frac{\kappa \cdot \varphi(t)}{2t} M_2(\mathbb{F}_{q^t}) \\ & \oplus \bigoplus_{\substack{t|m' \\ t \text{ even}}} \frac{2^{\nu-2}\kappa \cdot \varphi(t)}{t} M_2(\mathbb{F}_{q^{2t}}) \oplus \bigoplus_{\substack{t|m' \\ t \text{ odd} \\ t \neq 1}} \frac{(2^{\nu-1} - 1)\kappa \cdot \varphi(t)}{2t} M_2(\mathbb{F}_{q^{2t}}). \end{aligned}$$

$$\text{where } i = \begin{cases} 0 & \text{if } \nu_2(q+1) > \nu_2(\frac{n}{2}) \\ 1 & \text{if } \nu_2(q+1) \leq \nu_2(\frac{n}{2}). \end{cases}$$

Proof: First, we consider the case $n \not\equiv 3 \pmod{4}$ or $8 \nmid n$, so every irreducible factor of $x^n - 1$ is a binomial, and except for the factors $x - 1$ and $x + 1$, we have that any irreducible factor of the form $x^t - a$ is not auto-reciprocal. Thus, we have two cases to analyse:

- If n is odd, we have that $\epsilon = 0$ or 1 and $r = 1$. By Lemma 4.1 there exist $\frac{\kappa \cdot \varphi(t)}{t}$ irreducible factors of degree t and by Theorem 3.3 there exist two components isomorphic to \mathbb{F}_q , $\frac{\kappa \cdot \varphi(t)}{2t}$ components of the form $M_2(\mathbb{F}_{q^t})$ if $t > 1$ and $\frac{\kappa-1}{2}$ components of the form $M_2(\mathbb{F}_q)$ if $t = 1$, where t is a divisor of m . So we obtain item (1).
- If n is even, we have that $\epsilon = 2$ and there exist four components isomorphic to \mathbb{F}_q . In addition, every factor of $x^n - 1$ different that $x \pm 1$ is a non-auto-reciprocal binomial, then $r = 2$, and by the same argument of the previous case there exist $\frac{\kappa \cdot \varphi(t)}{2t}$ components of the form $M_2(\mathbb{F}_{q^t})$ if $t > 1$ and $\frac{\kappa-2}{2}$ components of the form $M_2(\mathbb{F}_q)$ if $t = 1$, where t is a divisor of m . So, we obtain item (2.1).

Finally, in the case which $q \equiv 3 \pmod{4}$ and $8|n$, every factor of $x^n - 1$ is a binomial or a trinomial. The unique auto-reciprocal factor of the form $x^t - a$ with t odd is $f_1 = x - 1$. Now, suppose that $x^{2t} - (\alpha^{ul'} + \alpha^{qul'})x^t + \theta^{ul'}$ is an irreducible factor of $x^n - 1$ as in Lemma 4.1 item (b), such that it is an auto-reciprocal polynomial. It follows that $\theta^{ul'} = 1$ and therefore $(q-1)|ul'$. Since

$$l' = \frac{q-1}{\gcd(n, q-1)} \cdot \frac{q+1}{2^\nu},$$

the polynomial is auto-reciprocal when $\gcd(n, q-1)|u \cdot \frac{q+1}{2^\nu}$ and we have two cases to consider:

- i) If $\nu_2(q+1) \leq \nu_2(\frac{n}{2})$ then $\frac{q+1}{2^\nu}$ is odd and $\gcd(\gcd(n, q-1), \frac{q+1}{2^\nu}) = 1$, therefore $\gcd(n, q-1)|u$. But $t|m'|n$ and $\gcd(t, u) = 1$, then these conditions imply that $t = 1$ and u is a multiple of $\gcd(n, q-1)$ not divisible by 2^ν and less than $2^\nu \gcd(n, q-1)$. So there exist $2^\nu - 2$ values of u that generate $2^{\nu-1} - 1$ auto-reciprocal factors, all of them of degree 2, each one generating a component of the form $M_2(\mathbb{F}_q)$. In addition, we have $\kappa - 2$ irreducible factors of degree 1, each one generating a component of the same type.

Therefore there exist $(\kappa - 2) + (2^{\nu-1} - 1) = \kappa + 2^{\nu-1} - 3$ components $M_2(\mathbb{F}_q)$ and

$$\frac{k}{4}(2^\nu - 1) - (2^{\nu-1} - 1) = 2^{\nu-2}\kappa - 2^{\nu-1} - \frac{k}{4} + 1$$

components $M_2(\mathbb{F}_{q^2})$.

- ii) If $\nu_2(q+1) > \nu_2(\frac{n}{2})$ then $\frac{q+1}{2^\nu}$ is even and $\gcd(\gcd(n, q-1), \frac{q+1}{2^\nu}) = 2$, therefore $\frac{1}{2}\gcd(n, q-1)|u$. Similarly, we obtain $t = 1$ and u is a multiple of $\frac{1}{2}\gcd(n, q-1)$ non divisible by 2^ν and less than $2^\nu \gcd(n, q-1)$. So there exist $2^{\nu+1} - 2$ values of u and then $2^\nu - 1$ auto-reciprocal factors, all of them of degree 2.

Then there exist $\kappa + 2^\nu - 3$ components $M_2(\mathbb{F}_q)$ and $2^{\nu-2}\kappa - 2^\nu - \frac{k}{4} + 1$ components $M_2(\mathbb{F}_{q^2})$. \square

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DEPARTAMENTO DE MATEMÁTICA, UNIVERSIDADE FEDERAL DE MINAS GERAIS, UFMG, BELO
HORIZONTE, MG, 30123-970, BRAZIL,
E-mail address: `fbrocher@mat.ufmg.br`